Dirac Monopoles

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Last dated on December 15, 2020
from November 26, 2020

Abstract

Up to now, there has not been any obvious evidence of the existence of Dirac monopoles. However, it would seem plausible to quantize electronic charge by supposing Dirac monopoles. Herein, one little attempt to figure out the underlying thoughts would be involved. Additionally, Yang's article *Magnetic monopoles, fiber bundles, and gauge fields* published on "Annals New York Academy of Sciences" would be the main reference, as well as Bjørn Felsager's *Geometry, Particles, and Fields* especially its chapter 9.

Contents

*Edited at Hunan University, according to Prof. Zhen FANG's advice and interest.

Part I
Mathematical Basis from Bjørn

1 Differentiable Manifolds-Tensor Analysis

1.1 Coordinate Systems

ϕ is smooth map, that all components have partial derivatives of arbitrary high order.

Definition 6.1 Let M be a subset of $\mathbb{R}^N$. A coordinate system on M is a pair (ϕ, U) where $U \subset \mathbb{R}^N$ is an open subset and $\phi : U \to M$ is an injective map with the following two properties:
a. ϕ is a homeomorphism from the open set U onto the open set $\phi(U)$. (ϕ respects the topology of M.)
b. ϕ is a smooth regular map. (ϕ respects the smoothness of M.)

Definition 6.2 Let M be a subset of $\mathbb{R}^N$. An atals on M is a family of coordinate systems $(\phi_i, U_i)_{i \in I}$ w/ the following properties:
a. The family $(\phi_i, U_i)_{i \in I}$ covers M; i.e., $M \subset \cup_{i \in I} \phi_i(U_i)$.
b. Any two coordinate systems in the family are smoothly related.

1.2 Differentiable Manifolds

Definition 6.3 f is differentiable at the point P_0 iff the Euclidean representative $\bar{f}$ is differentiable in the usual sense at $(x_0^1, \cdots, x_0^n)$.

Because an atlas makes it possible for us to introduce to concept of a differentiable map $f : M \to \mathbb{R}$, we say that an atlas generates a differentiable structure on M.

Definition 6.4 An atlas $(\phi_i, U_i)_{i \in I}$ on a subset M is called maximal if it has the following property: Whenever (ψ, V) is a coordinate system that is smoothly related to all the coordinates systems (ϕ_i, U_i) in our atlas, then (ψ, V) itself belongs to our atlas; i.e., $(\psi, V) = (\phi_j, U_j)$ for some $j \in J$.

Thus a maximal atlas comprises all possible smooth exchanges of the coordinates on M.

Definition 6.5 A differentiable structure on a Euclidean subset M is a maximal atlas on M.
Definition 6.6 A differentiable manifold M is a Euclidean subset M equipped w/ a differentiable structure.

Let M be a differentiable manifold. If we can cover it w/ a single coordinate system, we call it a simple manifold.

Any atlas on the sphere consists of at least two coordinate systems.

1.3 Product Manifolds and Manifolds Defined by Constraints

We define a coordinate system on the product set $M \times N$ as follows:

$$\phi \otimes \psi : U \times V \to M \times N$$
$$\phi \times \psi(x^1, \cdots, x^m; y^1, \cdots, y^n) = (\phi(x^1, \cdots, x^m); \psi(y^1, \cdots, y^n)).$$

If $(\phi_i, U_i)_{i\in I}$ is an atlas for M and $(\phi_j, U_j)_{j\in J}$ is an atlas for N, then $(\phi_i \otimes \psi_j, U_i \otimes V_j)$ is an atlas for $M \times N$. We call $M \times N$ the product manifold of M and N.

Theorem 6.1 (Theorem of implicit functions)
Let $f : \mathbb{R}^{m+n} \to \mathbb{R}^n$ be a smooth function. If $\bar{\bar{D}}_y f$ is a regular square matrix, when it is evaluated at a solution $(\bar{x}_0, \bar{y}_0)$ to the equation $f(\bar{x}, \bar{y}) = 0$, there exists a neighborhood U of $(\bar{x}_0, \bar{y}_0)$ and a smooth function $g : \mathbb{R}^m \to \mathbb{R}^n$ such that
1. $f(\bar{x}, \bar{y}) = 0$ iff $\bar{y} = g(\bar{x})$ for all $(\bar{x}, \bar{y})$ in the neighborhood U.
2. $\bar{\bar{D}}_x g = -(\bar{\bar{D}}_y f)^{-1}(\bar{\bar{D}}_x f)$.

Theorem 6.2 Let $f : \mathbb{R}^q \to \mathbb{R}^p$ be a smooth function $(q > p)$. Let M be the set of solutions to the equations $f(\bar{z}) = 0$. If M is nonempty and if f has maximal rank throughout M, then M is a differentiable manifold of dimension $(q - p)$.

1.4 Tangent Vectors

Let $\lambda : \mathbb{R} \to M$ be a smooth curve passing through P. Then λ generates a tangent vector at P, the velocity vector

$$\mathbf{v}_P = \frac{d\lambda}{dt}|_{t=0}$$

The set of vectors generated by all possible smooth curves in M passing through P forms a set of vectors called the tangent space and, it is denoted by

$$\mathbf{T}_P(M)$$

Consider the curves:

$$\lambda_1(t) = \phi(t, 0, 0, \cdots, 0); \cdots ; \lambda_n(t) = \phi(0, 0, \cdots, 0, t)$$

They are called the coordinate lines. By differentiation we now obtain the tangent vectors

$$\mathbf{e}_1 = \frac{d\lambda_1}{dt}|_{t=0}; \cdots ; \mathbf{e}_n = \frac{d\lambda_n}{dt}|_{t=0}$$

They are called the canonical frame vectors.

Definition 6.7 Let $\mathbf{v}_P$ be a tangent vector generated by λ and let λ have the coordinate representation $\bar{\lambda}(t) = (x^1(t), \cdots, x^n(t))$. Then the numbers $a^i = \frac{dx^i}{dt}$ are called the intrinsic coordinates of $\mathbf{v}_P$ with respect to the coordinate system (ϕ, U).

$$\mathbf{e}_{(2)j} = \mathbf{e}_{(1)i}\frac{\partial x^i}{\partial y^j}$$

A quantity that transform in the same way as the canonical frame vectors is said to transform covariantly (covarians = transform in the same way).

$$a^i_{(2)} = \frac{\partial y^i}{\partial x^j} a^j_{(1)}$$

A quantity that transforms with the reciprocal matrix $\frac{\partial y^i}{\partial x^j}$ is said to transform contravariantly (cotravarians = transforms in the opposite way).

We say that the vector field is smooth if its components $a^i(x^1, \cdots, x^n)$ are smooth functions of the coordinates.

1.5 Metrics

In $\mathbb{R}^n$ we have an inner product which is symmetric, bilinear, and positive definite.

$$\langle x|y\rangle = x^1y^1 + \cdots + x^ny^n$$

To each tangent space on the manifold M we therefore associate an inner product. This family of inner products is called a metric, and it is denoted by the letter $\mathbf{g}$. If $\mathbf{v}_P$ and $\mathbf{u}_P$ are tangent vectors belonging to the same tangent space $\mathbf{T}_P(M)$, we denote the value of their inner product by $\mathbf{g}(\mathbf{v}_P; \mathbf{u}_P)$.

$$g_{ij} = \mathbf{g}(\mathbf{e}_i; \mathbf{e}_j)$$

Definition 6.8 Let M be a differentiable manifold. A smooth metric $\mathbf{g}$ is a family of inner products defined on each of the tangent spaces, w/ the following properties:
a. $\mathbf{g}$ is a bilinear map that is symmetric and non-degenerate.
b. The metric coefficients g_{ij} depend smoothly upon the coordinates $(x^1, \cdots, x^n)$.

Lemma 6.1 g has the constant sign throughout M.

Definition 6.9
a. If all the eigenvalues are strictly positive, we say that the metric is a Euclidean metric. In that case the metric is positive definite. A manifold w/ a Euclidean metric is called a Riemannian manifold. b. If one of the eigenvalues is negative and the rest of them are positive, we say that the metric is a Minkowski metric. In this case the metric in non-definite. (a pseudo-Riemannian manifold)

Lemma 6.2 Let M be a manifold w/ a Minkowski metric. Let $\mathbf{T}_P(M)$ be the tangent space at the point P. Then we can decompose $\mathbf{T}_P(M)$ into an orthogonal sum

$$\mathbf{T}_P(M) = V_- \oplus U_+$$

where V_- is a one-dimensional subspace such that the restriction of $\mathbf{g}$ to V_- is negative definite and U_+ is an (n-1)-dimensional subspace such that the restriction of $\mathbf{g}$ to U_+ is positive definite.

Definition 6.10 Let M be a differentiable manifold w/ a Minkowski metric:
1. A vector w/ negative square is called Time-like.
2. A vector w/ zero square is called null vector.
3. A vector w/ positive square is called space-like.

1.6 The Minkowski Space

Theorem 6.3 A Poincaré transformation connecting two inertial frames is given by the formula

$$\bar{\bar{y}}_1 = \bar{\bar{A}}\bar{\bar{x}}_1 + \bar{\bar{b}}_1$$

where $\bar{\bar{A}}$ satisfies

$$\bar{\bar{A}}^\dagger \bar{\bar{\eta}} \bar{\bar{A}} = \bar{\bar{A}} \bar{\bar{\eta}} \bar{\bar{A}}^\dagger = \bar{\bar{\eta}}$$

Definition 6.12 A matrix $\bar{\bar{A}}$ w/ the property

$$\bar{\bar{A}}^\dagger \bar{\bar{\eta}} \bar{\bar{A}} = \bar{\bar{A}} \bar{\bar{\eta}} \bar{\bar{A}}^\dagger = \bar{\bar{\eta}}$$

is called a Lorentz-matrix. The Lorentz matrices constitute a group called the Lorentz group, which we denote by O(3,1). A homogeneous transformation

$$\bar{\bar{y}}_1 = \bar{\bar{A}}\bar{\bar{x}}_1$$

where $\bar{\bar{A}}$ is a Lorentz matrix, is called a Lorentz transformation.

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arccos\dfrac{z}{\sqrt{x^2+y^2+z^2}} \\ \varphi = \arctan\left(\dfrac{y}{x}\right) \end{cases}$$

$$\bar{\bar{D}}_{21} = \frac{\partial y^\alpha}{\partial x^\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x/r & y/r & z/r \\ 0 & \frac{-xz}{r^2\sqrt{r^2-z^2}} & \frac{-yz}{r^2\sqrt{r^2-z^2}} & \frac{z^2-r^2}{r^2\sqrt{r^2-z^2}} \\ 0 & -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix} \tag{6.34}$$

$$\bar{\bar{D}}_{12} = \frac{\partial x^\alpha}{\partial y^\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ 0 & \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ 0 & \cos\theta & -r\sin\theta & 0 \end{pmatrix} \tag{6.35}$$

1.7 The Action Principle for a Relativistic Particle

Lemma 6.3 Let γ be a geodesic:
1. Then γ extremizes the functional

$$I = \int_{\lambda_1}^{\lambda_2} \frac{1}{2} g_{\alpha\beta}(x(\lambda))\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}d\lambda$$

2. Furthermore, when $x^\alpha = x^\alpha(\lambda)$ extremizes the functional I, then

$$g_{\alpha\beta}(x(\lambda))\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$$

is const. along the geodesic, so that we can identify the parameter λ w/ the arc-length s.

1.8 Covectors

Definition 6.13 A covector ω_P at the point P is a linear map

$$\omega_P : \mathbf{T}_P(M) \to \mathbb{R}$$

$$\omega(\mathbf{v}_P) = \langle\omega|\mathbf{v}_P\rangle$$

So the covectors at the point P form a vector space, which mathematicians call the dual vector space and which they denote by

$$\mathbf{T}^*_P(M)$$

We can then define coordinates of the covector in the following way :

$$\omega_i = \langle\omega_P|\mathbf{e}_i\rangle$$

$$\mathbf{d}x^1 : \left(\frac{\partial x^1}{\partial x^1}, \frac{\partial x^1}{\partial x^2}, \cdots, \frac{\partial x^1}{\partial x^n}\right) = (1, 0, \cdots, 0)$$

The covectors $\mathbf{d}x^1, \cdots, \mathbf{d}x^n$ serve as the canonical frame vectors in the dual space $\mathbf{T}^*_P(M)$.

$$\mathbf{d}f = \frac{\partial \bar{f}}{\partial x^i}\mathbf{d}x^i$$

The symbol $\mathbf{d}$ is now interpreted as an exterior derivative, like the "old" formulas involving differentials. So in modern geometry we work w/ the same formulas; they have just been given a new interpretation.

1.9 Tensors

Definition 6.14 A cotensor $\mathbf{F}_P$ is a multilinear map

$$\mathbf{F}_P : \mathbf{T}_P(M) \times \cdots \times \mathbf{T}_P(M) \to \mathbb{R}$$

If $\mathbf{F}_P$ has k arguments, we say that $\mathbf{F}_P$ is a contensor of rank k, and the set of all cotensors of rank k will be denoted by $\mathbf{T}^{(0,k)}(M)$.

Consider a cotensor $\mathbf{F}_P$ of rank 2. Let it have the components F_{kl}. Then we can construct another cotensor of rank 2

$$F_{ij}\mathbf{d}x^i \otimes \mathbf{d}x^j$$

and the components of this cotensor:

$$F_{ij}\mathbf{d}x^i \otimes \mathbf{d}x^j(\mathbf{e}_k; \mathbf{e}_l) = F_{ij}\mathbf{d}x^i(\mathbf{e}_k) \cdot \mathbf{d}x^j(\mathbf{e}_l) = F_{ij}\delta^i_k\delta^j_l = F_{kl}$$

Hence $F_{ij}\mathbf{d}x^i \otimes \mathbf{d}x^j$ has the components F_{kl} too, and we conclude

$$\mathbf{F}_P = F_{ij}\mathbf{d}x^i \otimes \mathbf{d}x^j$$

Definition 6.15 A mixed tensor $\mathbf{F}_P$ of type (k, l) is a multilinear map

$$\mathbf{F}_P : \underbrace{\mathbf{T}^*_P(M) \times \cdots \times \mathbf{T}^*_P(M)}_{k} \times \underbrace{\mathbf{T}_P(M) \times \cdots \times \mathbf{T}_P(M)}_{l} \to \mathbb{R}$$

A mixed tensor of type (k, l) has rank $k + l$, and the space of all mixed tensors of type (k, l) is denoted $\mathbf{T}^{(k,l)}_P(M)$.

1.10 Tensor Fields in Physics

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

In the inertial frame the electric field strength $\mathbf{E}$ vanishes, and

$$\mathbf{B} = \frac{g}{4\pi} \cdot \frac{\mathbf{r}}{r^3}$$

where g is the strength of the monopole. Therefore:

$$\bar{\bar{F}}_{(1)\alpha\beta} = \frac{g}{4\pi r^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & +z/r & -y/r \\ 0 & -z/r & 0 & +x/r \\ 0 & +y/r & -x/r & 0 \end{pmatrix}$$

According to eq. 6.34 & 6.35

$$\bar{\bar{F}}_{(2)} = \bar{\bar{D}}^{\dagger}_{(12)}\bar{\bar{F}}_{(1)}\bar{\bar{D}}_{(12)}$$

$$\bar{\bar{F}}_{(2)} = \frac{g}{4\pi}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin\theta \\ 0 & 0 & -\sin\theta & 0 \end{pmatrix} \tag{6.79}$$

If we have a tensor $\mathbf{T}$ of rank 2, we can form the following two invariants:
a. The trace of $\mathbf{T}$: T^{α}_{α}
b. The "square" of $\mathbf{T}$: $\frac{1}{2}T_{\alpha\beta}T^{\alpha\beta}$.

2 Differential Forms and the Exterior Calculus

2.1 Introduction

You can't compare two tensor w/ different base points P and Q just by comparing their components!

2.2 k-Forms-The Wedge Product

We say that $\mathbf{F}$ is skew-symmetric if it changes sign whenever we exchange two of its arguments:

$$\mathbf{F}(\cdots;\mathbf{u}_P;\cdots;\mathbf{v}_P;\cdots) = -\mathbf{F}(\cdots;\mathbf{v}_P;\cdots;\mathbf{u}_P;\cdots)$$

Definition 7.1 A k-form is a skew-symmetric cotensor of rank k.

Definition 7.2 Let $\mathbf{F}$ be a k-form w/ components $F_{a_1\cdots a_k}$ and $\mathbf{G}$ an m-form w/ components $G_{b_1\cdots b_m}$. Then the wedge product $\mathbf{F}\wedge\mathbf{G}$ is the $(k+m)$-form w/ components

$$\frac{(k+m)!}{k!m!}F_{[a_1\cdots a_k}G_{b_1\cdots b_m]}$$

$$2A_{[a}B_{b]} = A_aB_b - A_bB_a = A_aB_b - B_aA_b$$
$$\mathbf{A}\wedge\mathbf{B} = [\mathbf{A}\otimes\mathbf{B} - \mathbf{B}\otimes\mathbf{A}]$$

Theorem 7.1 The wedge product is associative and distributive:

$$(\mathbf{F}\wedge\mathbf{G})\wedge\mathbf{H} = \mathbf{F}\wedge(\mathbf{G}\wedge\mathbf{H})$$

If $\mathbf{F}$ is a k-form and $\mathbf{G}$ is an m-form, then

$$\mathbf{F} \wedge \mathbf{G} = (-1)^{km} \mathbf{G} \wedge \mathbf{F}$$

Theorem 7.2 If $\mathbf{F}$ has the components $F_{i_1 \cdots i_k}$, we can expand $\mathbf{F}$ in the following way:

$$\mathbf{F} = \frac{1}{k!} F_{i_1 \cdots i_k} \mathbf{d}x^{i_1} \wedge \cdots \wedge \mathbf{d}x^{i_k}$$

2.3 The Exterior Derivative

Definition 7.3 If $\mathbf{F}$ is a differential form of degree k characterized by the components $F_{i_1 \cdots i_k}(x)$, then $\mathbf{dF}$ is the differential form of degree $k+1$ characterized by the components

$$\frac{(k+1)!}{k!} \partial_{[\mu} F_{i_1 \cdots i_k]}$$

The differential operator $\mathbf{d} : N^k(M) \to N^{k+1}(M)$ is called the exterior derivative.

Theorem 7.3 (Poincaré's lemma) automatically vanishes if applied twice.

$$\mathbf{dd} = 0$$

Theorem 7.4 (Leibniz's rule for differential forms) If $\mathbf{F}$ is a k-form and $\mathbf{G}$ is an l-form, then

$$\mathbf{d}(\mathbf{F} \wedge \mathbf{G}) = \mathbf{dF} \wedge \mathbf{G} + (-1)^k \mathbf{F} \wedge \mathbf{dG}$$

Definition 7.4
a. A differential form $\mathbf{F}$ is closed if its exterior derivative vanishes; i.e., if $\mathbf{dF} = 0$.
b. A differential form $\mathbf{G}$ is exact if it's the exterior derivative of another differential form $\mathbf{F}$; i.e., $\mathbf{G} = \mathbf{dF}$.

Definition 7.5 A k-form ω is called simple if there exists k smooth scalar fields $\phi_1, \cdots, \phi_k$ such that ω can be decomposed as

$$\omega = \mathbf{d}\phi_1 \wedge \cdots \wedge \mathbf{d}\phi_k$$

Lemma 7.1 A simple form is closed and exact.

2.4 The Volume Form

$$\mathrm{Vol}(\mathbf{v}_1; \cdots; \mathbf{v}_n) = \det[\bar{\bar{B}}] \tag{7.21}$$

where the matrix elements B^i_j are the coordinates of $\mathbf{v}_j$

$$\mathbf{v}_j = \mathbf{e}^0_i B^i_j$$

Let $(\mathbf{u}_1, \cdots, \mathbf{u}_n)$ be an n-tuple of tangent vectors, where $\mathbf{u}_j$ is characterized by the contravariant components A^i_j.

$$\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n(\mathbf{u}_1, \cdots, \mathbf{u}_n) = \det(\bar{\bar{A}})$$

$$\mathbf{u}_j = \mathbf{e}_i A^i_j = \mathbf{e}^0_k E^k_i A^i_j$$
$$\mathrm{Vol}(\mathbf{u}_1, \cdots, \mathbf{u}_n) = \det(\bar{\bar{E}}\bar{\bar{A}}) = \det(\bar{\bar{E}}) \cdot \mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n(\mathbf{u}_1, \cdots, \mathbf{u}_n) \tag{7.23}$$

It remains to interpret $\det[\bar{\bar{E}}]$. Consider the coordinate exchange

$$y^i = E^i_j x^j$$

then the transformation formula for the metric coefficients may be rearranged as

$$g_{ij} = g^0_{kl} E^k_i E^l_j = \delta_{kl} E^k_i E^l_j = E^k_i E^k_j$$

i.e.,

$$\bar{\bar{G}} = \bar{\bar{E}}^\dagger \bar{\bar{E}}$$

It follows that

$$g = \det[\bar{\bar{G}}] = \left[\det(\bar{\bar{E}})\right]^2$$

Inserting this into (7.23), we have thus determined the volume form in the tangent space $\mathbf{T}_P(M)$:

$$\sqrt{g}\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n$$

Definition 7.6 A manifold M is said to be orientable if we can choose an atlas $(\phi_i, U_i)_{i \in I}$ such that any two coordinate systems in our atlas generate the same orientation; i.e.

$$\det\left(\frac{\partial x^\alpha}{\partial y^\beta}\right)$$

is positive throughout all overlapping regions.

Lemma 7.2 On a manifold w/ a Minkowski metric, $\sqrt{-g(x)}$ transforms according to the rule

$$\sqrt{-g_{(2)}(y)} = \left|\det\left(\frac{\partial x^\alpha}{\partial y^\beta}\right)\right| \sqrt{-g_{(1)}(x)}$$

(On a manifold w/ a Euclidean metric we just replace $\sqrt{-g}$ by $\sqrt{g}$.)

2.5 The Dual Map

Definition 7.8 Let $\mathbf{F}$ be a k-form on an orientable manifold M^n w/ a metric $\mathbf{g}$. Then the dual form $^*\mathbf{F}$ is the $(n-k)$-form characterized by the components

$$^*\mathbf{F}_{a_1 \cdots a_{n-k}} = \begin{cases} \dfrac{1}{k!}\sqrt{g(x)}\epsilon_{a_1 \cdots a_{n-k} b_1 \cdots b_k} F^{b_1 \cdots b_k} & \text{Euclidean metric} \\ \dfrac{1}{k!}\sqrt{-g(x)}\epsilon_{a_1 \cdots a_{n-k} b_1 \cdots b_k} F^{b_1 \cdots b_k} & \text{Minkowski metric} \end{cases} \tag{7.38}$$

Theorem 7.5

$$\begin{aligned} &^{**}\mathbf{T} = (-1)^{k(n-k)}\mathbf{T} && \text{Euclidean metric} \\ &^{**}\mathbf{T} = -(-1)^{k(n-k)}\mathbf{T} && \text{Minkowski metric} \end{aligned} \tag{7.43}$$

Theorem 7.6 Let $\mathbf{A}$ and $\mathbf{B}$ be 1-forms. Then

$$\begin{array}{ll} {}^*({}^*\mathbf{A}\wedge\mathbf{B}) = {}^*({}^*\mathbf{B}\wedge\mathbf{A}) = A^iB_i & \text{Euclidean metric} \\ {}^*({}^*\mathbf{A}\wedge\mathbf{B}) = {}^*({}^*\mathbf{B}\wedge\mathbf{A}) = -A^\mu B_\mu & \text{Minkowski metric} \end{array}$$

Theorem 7.7 Let $\mathbf{T}$ and $\mathbf{U}$ be k-forms. Then

$$\begin{array}{ll} {}^*({}^*\mathbf{T}\wedge\mathbf{U}) = \langle\mathbf{T}|\mathbf{U}\rangle = \frac{1}{k!}T^{i_1\cdots i_k}U_{i_1\cdots i_k} & \text{Euclidean metric} \\ -{}^*({}^*\mathbf{T}\wedge\mathbf{U}) = \langle\mathbf{T}|\mathbf{U}\rangle = \frac{1}{k!}T^{\mu_1\cdots\mu_k}U_{\mu_1\cdots\mu_k} & \text{Minkowski metric} \end{array}$$

Theorem 7.8
a. Let M be a Riemannian orientable mfd. Then the scalar product $(\mathbf{T},\mathbf{U})\to\langle\mathbf{T}|\mathbf{U}\rangle$ defined a Euclidean metric.
b. Let M be a pseudo-Riemannian orientable mfd. Then the scalar product $(\mathbf{T},\mathbf{U})\to\langle\mathbf{T}|\mathbf{U}\rangle$ defined an indefinite metric.

Theorem 7.9
a. Let M be a Riemannian orientable manifold. The dual map $*$ is an isometry for all k-forms $\mathbf{T}$ and $\mathbf{U}$.

$$\langle{}^*\mathbf{T}|{}^*\mathbf{U}\rangle = \langle\mathbf{T}|\mathbf{U}\rangle$$

b. Let M be an orientable manifold w/ a Minkowski metric. The dual map $*$ is an antiisometry for all k-forms $\mathbf{T}$ and $\mathbf{U}$.

$$\langle{}^*\mathbf{T}|{}^*\mathbf{U}\rangle = -\langle\mathbf{T}|\mathbf{U}\rangle$$

2.6 The Codifferential and the Laplacian

Definition 7.9 For an arbitrary k-form $\mathbf{T}$, the codifferential of $\mathbf{T}$, denoted by $\delta\mathbf{T}$, is the $(k-1)$-form defined by

$$\delta\mathbf{T} = \begin{cases} +(-1)^{k(n-k+1)}{}^*\mathbf{d}^*\mathbf{T} & \text{Euclidean metric} \\ -(-1)^{k(n-k+1)}{}^*\mathbf{d}^*\mathbf{T} & \text{Minkowski metric} \end{cases} \tag{7.59}$$

Theorem 7.10

$$\delta^2 = 0$$

$$\delta\delta = \pm({}^*\mathbf{d}^*)({}^*\mathbf{d}^*) = \pm{}^*\mathbf{dd}^* = 0$$

Definition 7.10

$$-\Delta = \delta\mathbf{d} + \mathbf{d}\delta$$

Defition 7.11
a. A differential form $\mathbf{F}$ is coclosed if its codifferential vanishes; $\delta\mathbf{F} = 0$
b. A differential form $\mathbf{G}$ is coexact if it's the codifferential of another differential form $\mathbf{F}$; $\mathbf{G} = \delta\mathbf{F}$

Definition 7.12 A differnetial form $\mathbf{T}$ is called harmonic if $\Delta\mathbf{T} = 0$

Definition 7.13 A differential form $\mathbf{T}$ is called primitively harmonic if it's both closd and coclosed; i.e., $\mathbf{dT} = \delta\mathbf{T} = 0$

2.7 Exterior Calculus in 3 and 4 Dimensions

$$^*\mathbf{G} = \frac{1}{3!}\sqrt{g}\epsilon_{ijk}G^{ijk}$$

	Forms	Components
Scalar product	$\langle A\vert B\rangle$	A_iB^i
Contraction	$F \cdot A$	$F_{ij}A^j$
Wedge product	$A \wedge B$	$A_iB_j - A_jB_i$
Wedge product	$F \wedge A$	$F_{ij}A_k + F_{jk}A_i + F_{ki}A_j$

Conventional vector analysis	Exterior algebra
$\mathbf{a}\cdot\mathbf{b}$	$\langle A\vert B\rangle$
$\mathbf{a}\times\mathbf{b}$	$A \wedge B$
$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$	$(A \wedge B) \wedge C$
$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$	$-(A \wedge B)\cdot C$
$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b} = (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}$	$A\wedge B\wedge C = C\wedge A\wedge B = B\wedge C\wedge A$
$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - \mathbf{a}(\mathbf{b}\cdot\mathbf{c})$	$(A\wedge B)\cdot C = A\langle B\vert C\rangle - \langle A\vert C\rangle B$

$$\mathbf{d}^*\mathbf{A} = \partial_i(^*\mathbf{A})_{jk} + \partial_j(^*\mathbf{A})_{ki} + \partial_k(^*\mathbf{A})_{ij}$$

$$(\mathbf{d}^*\mathbf{A})_{123} = \partial_1(\sqrt{g}A^1) + \partial_2(\sqrt{g}A^2) + \partial_3(\sqrt{g}A^3)$$

$$^*(\mathbf{d}^*\mathbf{A}) = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}A^i)$$

$$-\delta\mathbf{A} = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}A^i)$$

$$\mathbf{d}(\mathbf{F}\wedge\mathbf{G}) = \mathbf{dF}\wedge\mathbf{G} + (-1)^{\deg\mathbf{F}}\mathbf{F}\wedge\mathbf{dG} \tag{7.17}$$

Note: P380 & P383

2.8 Electromagnetism and the Exterior Calculus

Conventional vector analysis	Exterior calculus
$\nabla\cdot\mathbf{B} = 0$	$\mathrm{dB} = 0$
$\frac{\partial\mathbf{B}}{\partial t} + \nabla\times\mathbf{E} = \mathbf{0}$	$\frac{\partial\mathbf{B}}{\partial t} + \mathbf{dE} = 0$
$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}$	$-\delta\mathbf{E} = \frac{\rho}{\epsilon_0}$
$\frac{\partial\mathbf{E}}{\partial t} - c^2\nabla\times\mathbf{B} = -\frac{\mathbf{j}}{\epsilon_0}$	$\frac{\partial\mathbf{E}}{\partial t} - c^2\delta\mathbf{B} = -\frac{\mathbf{J}}{\epsilon_0}$

Part II
Yang's review

3 Magnetic monopoles and nontrivial bundles

The magnetic monopole is the magnetic charge.

In particular, with the existence of a magnetic monopole of strength g, electric charges and magnetic charges must necessarily be quantized, in quantum mechanics.

Theorem: Consider a magnetic monopole of strength $g \neq 0$ at the origin, and consider a sphere of radius R around the origin. There does not exist a vector potential $\mathbf{A}$ for the monopole magnetic field that is singularity free on the sphere.

Suppose that there was a singularity-free $\mathbf{A}$, and consider the loop integral according to Stoke's theorem: (Figure 1)

$$\oint A_\mu dx^\mu = \Omega_\alpha = \Omega_\beta$$

where Ω_α and Ω_β are the total upward magnetic fluxes through caps α and β. However, similar to electric charge, this value should be $4\pi g \neq 0$, denoting a contradiction and proving the theorem.

Following the idea from *Differential Geometry*, we could describe the whole globe with a double system of coordinates from up and down of the globe, respectively. (Figure 2.)

$$(A_r)^a = (A_\theta)^a = 0 \qquad (A_r)^b = (A_\theta)^b = 0$$
$$(A_\phi)^a = +\frac{g}{r\sin\theta}(1-\cos\theta)$$
$$(A_\phi)^b = -\frac{g}{r\sin\theta}(1+\cos\theta)$$

According to eq. (6.79), we can decompose the monopole field as

$$\boxed{\mathbf{F} = \frac{g}{4\pi}\sin\theta\mathbf{d}\theta\wedge\mathbf{d}\varphi = \frac{-g}{4\pi}\mathbf{d}(\cos\theta)\wedge\mathbf{d}\varphi} \tag{7.89}$$

Thus we can use the gauge potential

$$\mathbf{A} = \frac{-g}{4\pi}\cos\theta\mathbf{d}\varphi \tag{7.90}$$

and it can't define a smooth vector field throughout our manifold, since the coordinate system itself is singular! (z-axis).

$$\mathbf{A} = \frac{-g}{4\pi}\left[-\frac{yz}{r(x^2+y^2)}\mathbf{d}x + \frac{xz}{r(x^2+y^2)}\mathbf{d}y\right]$$

$\mathbf{A}$ is highly singular on the z-axis in accordance with our general result: **There can exist no glocal gauge potential generating the monopole field.**

The singular gauge potential $\mathbf{A} = \frac{-g}{4\pi}\cos\theta\mathbf{d}\varphi$ is obviously not the only gauge potential generating $\mathbf{F}$. Observe that $\mathbf{d}(\mathbf{d}\varphi) = 0$, consider

$$\mathbf{A}' = \mathbf{A} + \frac{g}{4\pi}\mathbf{d}\varphi = \frac{-g}{4\pi}\cos\theta\mathbf{d}\varphi + \frac{g}{4\pi}\mathbf{d}\varphi$$

Although it formally looks like a gauge transformation, it's a singular gauge transformation because φ is a singular coordinate function. Nevertheless, $\mathbf{A}'$ generates the same field strengths.

$$\mathbf{A}' = \frac{-g}{4\pi}(\cos\theta - 1)\mathbf{d}\varphi$$

and that the coefficient $\cos\theta - 1$ vanishes on the positive z-axis. This suggests that $\mathbf{A}'$ is singular only on the negative z-axis.

Thus we are left with an semi-infinite Dirac string : The negative z-axis.

In the region of overlap, since both of the two sets of vector potentials share the same curl, the difference between them must be curl-less and therefore must be a gradient.

$$\mathbf{A}' \to \mathbf{A} + \nabla\chi$$

$$\nabla \cdot \mathbf{B}' = \nabla \cdot (\nabla \times \mathbf{A}') = \nabla \cdot [\nabla \times (\mathbf{A} + \nabla\chi)] = \nabla \cdot (\nabla \times \mathbf{A}) = \nabla \cdot \mathbf{B} = 0$$

Indeed a simple calculation shows: (where ϕ is the azimuthal angle.)

$$(A_\mu)^a - (A_\mu)^b = \partial_\mu \alpha \qquad \text{where} \quad \alpha = 2g\phi$$

The Schrödinger eq $\hat{\mathbf{H}}\Psi = E\Psi$ for an electron in the monopole field is thus:

$$\frac{1}{2m}(p - eA_a)^2 \psi_a + V\psi_a = E\psi_a \qquad (R_a)$$
$$\frac{1}{2m}(p - eA_b)^2 \psi_b + V\psi_b = E\psi_b \qquad (R_b)$$

According to U(1) symmetry:

$$\psi_a = S\psi_b \qquad S = \exp(ie\alpha)$$

or

$$\psi_a = [\exp(2iq\phi)]\,\psi_b \qquad q = eg$$

Around the equator which is entirely in R_a, ψ_a is single valued. Therefore, S must return to its original value when one goes around the equator, and Dirac's quantization condition is:

$$2q = 2eg = \text{integer}$$

That Two phases satisfy the same condition of transition in the overlap region, are called a section in math. We see that around a monopole the electron wave function is a section and not an ordinary function.

4 Hilbert space of sections

Physics	Mathematics
Special Relativity	4-dimensional space-time
General Relativity	Riemannian Geometry
Quantum Mechanics	Hilbert Space
Electromagnetism and Non-Abelian Gauge Fields	Fiber Bundles

We will now attempt to construct angular momentum operators. Define:

$$\mathbf{L} = \mathbf{r} \times (\mathbf{p} - e\mathbf{A}) - \frac{q\mathbf{r}}{r}$$

5 Monopole harmonics

$$L^2 Y_{q,l,m} = l(l+1) Y_{q,l,m}$$
$$L_z Y_{q,l,m} = m Y_{q,l,m}$$

$$\int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} \left|Y_{q,l,m}\right|^2 d\phi = 1$$

6 Early history of the concept of gauge fields

Gauge Field Terminology	Bundle Terminology
Gauge (or global gauge)	principal coordinate bundle
Gauge type	principal fiber bundle
Gauge potential b_{μ}^{k}	connection on a principal fiber bundle
$S = \exp(ie\alpha)$	transition function
Phase factor Φ_{OP}	parallel displacement
Field strength $f_{\mu\nu}^{k}$	curvature
Source (electric) J_{μ}^{k}	?
Electromagnetism	connection on a U(1) bundle
Isotopic spin gauge field	connection on a SU(2) bundle
Dirac's monopole quantization	classification of U(1) bundle according to first Chern class
Electromagnetism w/o monopole	connection on a trivial U(1) bundle
Electromagnetism w/ monopole	connection on a non-trivial U(1) bundle

$$\Phi_{BA} = \exp\left[-i(e/hc)\int_A^B A_\mu dx^\mu\right]$$

7 Extension: Chern Number

How to describe topological photonics?
https://www.zhihu.com/question/53921746/answer/954310741

Berry connection

$$\mathcal{A}(\mathbf{k}) = \langle u(\mathbf{k})|i\nabla_{\mathbf{k}}|u(\mathbf{k})\rangle$$

Berry phase

$$\oint \mathcal{A}(\mathbf{k}) \cdot d\mathbf{l}$$

Berry curvature

$$\mathcal{F}(\mathbf{k}) = \nabla_{\mathbf{k}} \times \mathcal{A}(\mathbf{k})$$

Berry flux

$$\iint \mathcal{F}(\mathbf{k}) \cdot d\mathbf{s}$$

Chern number

$$C = \frac{1}{2\pi} \oiint \mathcal{F}(\mathbf{k}) \cdot d\mathbf{s}$$

The total magnetic flux is none other than the first Chern number of the principal bundle, and depends only upon the choice of the principal bundle, and not the specific connection over it. In other words, it is a topological invariant.
https://en.wikipedia.org/wiki/Magnetic_monopole

Berry connection and curvature
https://en.wikipedia.org/wiki/Berry_connection_and_curvature

Chern class
https://en.wikipedia.org/wiki/Chern_class

Chern number and winding number, differentiation and similarity
https://www.zhihu.com/question/263642742/answer/271983200

The family of Hall effect
https://zhuanlan.zhihu.com/p/194237287

Dirac non-integrable phase factors and monopole
https://zhuanlan.zhihu.com/p/34644618